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Hochschild cohomology ring of an order of a quaternion algebra

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Introduction

The cohomology theory of associative algebras was initiated by Hochschild [6], Cartan and Eilenberg [1] and MacLane [7]. Let R be a commutative ring and Λ an R-algebra which is a finitely generated projective R-module. If M is a Λ-bimodule (i.e., a $\Lambda^{e}=\Lambda\otimes_{R}\Lambda^{\mathrm{op}}$-module), then the nth Hochschild cohomology of Λ with coefficients in M is defined by $H^{n}(\Lambda, M):=\mathrm{Ext}_{\Lambda^{e}}^{n}(\Lambda, M)$. We set $HH^{n}(\Lambda)=H^{n}(\Lambda,\Lambda)$. The cup product gives $HH^{*}(\Lambda):=\bigoplus_{n\geq 0}HH^{n}(\Lambda)$ a graded ring structure with $1\in Z\Lambda\simeq HH^{0}(\Lambda)$ where $Z\Lambda$ denotes the center of Λ. $HH^{*}(\Lambda)$ is called the Hochschild cohomology ring of Λ. It is known that the cup product coincides with the Yoneda product on the Ext-algebra. Note that the Hochschild cohomology ring $HH^{*}(\Lambda)$ is graded-commutative, that is, for $\alpha\in HH^{p}(\Lambda)$ and $\beta\in HH^{q}(\Lambda)$ we have $\alpha\beta=(-1)^{pq}\beta\alpha$. The Hochschild cohomology is an important invariant of algebras, however the Hochschild cohomology ring is difficult to compute in general.

Let G denote the generalized quaternion 2-group of order 2^{r+2} for $r\geq 1$:

$$Q_{2^{r}}=\langle x,y \mid x^{2^{r+1}}=1, x^{2^{r}}=y^{2}, yxy^{-1}=x^{-1}\rangle.$$

We set $e=(1-x^{2^{r}})/2\in\mathbb{Q}G$ and denote xe by ζ, a primitive 2^{r+1}-th root of e. Then e is a centrally primitive idempotent of $\mathbb{Q}G$. The simple component $\mathbb{Q}Ge$ is just the ordinary quaternion algebra over the field $K:=\mathbb{Q}(\zeta+\zeta^{-1})$ with identity e, that is, $\mathbb{Q}Ge=K\oplus Ki\oplus Kj\oplus Kij$ where we set $i=x^{2^{r-1}}e$ and $j=ye$ (see [2, (7.40)]). Note that $\zeta^{k}j=j\zeta^{-k}$ and $\zeta^{2^{r}}=-e$ hold. In the following we set $R=\mathbb{Z}[\zeta+\zeta^{-1}]$, the ring of integers of K, and we set $\Gamma=\mathbb{Z}Ge=R\oplus R\zeta\oplus Rj\oplus R\zeta j$. Note that R is a commuting parameter ring, because y commutes with $x+x^{-1}$. Then Γ is an R-order of $\mathbb{Q}Ge$. In particular if $r=1$, $\Gamma=\mathbb{Z}e\oplus\mathbb{Z}i\oplus\mathbb{Z}j\oplus\mathbb{Z}ij$ is just the ordinary quaternion algebra over $\mathbb{Z}$ with identity e.

We will give an efficient bimodule projective resolution of Γ, and we will determine the ring structure of the Hochschild cohomology $HH^{*}(\Gamma)$ by calculating the Yoneda products using this bimodule projective resolution. This paper is a summary of [3].

1 A bimodule projective resolution of Γ

In this section, we state a Γ^{e}-projective resolution of Γ.

In general, $\Gamma\otimes\Gamma$ is a left Γ^{e}-module (i.e., a Γ-bimodule) by putting

$$(a\otimes b^{\circ})\cdot(\gamma_{1}\otimes\gamma_{2}):=a\gamma_{1}\otimes\gamma_{2}b$$

for all $a, b, \gamma_1, \gamma_2 \in \Gamma$. For each $q \geq 0$, let Y_q be a direct sum of $q+1$ copies of $\Gamma \otimes \Gamma$. As elements of Y_q, we set

$$c_q^s = \begin{cases} (0, \dots, 0, \underbrace{e \otimes e}_{s}, 0, \dots, 0) & (\text{if } 1 \leq s \leq q+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have $Y_q = \bigoplus_{k=1}^{q+1} \Gamma c_q^k \Gamma$. Let $t = 2^r$. Define left Γ^e-homomorphisms $\pi : Y_0 \to \Gamma; c_0^1 \mapsto e$ and $\delta_q : Y_q \to Y_{q-1}$ $(q > 0)$ given by

$$\delta_q(c_q^s) = \begin{cases} -\zeta c_{q-1}^s + c_{q-1}^s \zeta + (-1)^{(q-s)/2} \zeta j c_{q-1}^{s-1} j \zeta - c_{q-1}^{s-1} & \text{for } q \text{ even, } s \text{ even,} \\ \displaystyle\sum_{l=0}^{t-1} \zeta^{t-1-l} c_{q-1}^s \zeta^l + (-1)^{(q-s-1)/2} j c_{q-1}^{s-1} j + c_{q-1}^{s-1} & \text{for } q \text{ even, } s \text{ odd,} \\ \displaystyle -\sum_{l=0}^{t-1} \zeta^{t-1-l} c_{q-1}^s \zeta^l + (-1)^{(q-s-1)/2} j c_{q-1}^{s-1} j - c_{q-1}^{s-1} & \text{for } q \text{ odd, } s \text{ even,} \\ \zeta c_{q-1}^s - c_{q-1}^s \zeta + (-1)^{(q-s)/2} \zeta j c_{q-1}^{s-1} j \zeta + c_{q-1}^{s-1} & \text{for } q \text{ odd, } s \text{ odd.} \end{cases}$$

Theorem 1. *The above (Y, π, δ) is a Γ^e-projective resolution of Γ.*

Proof. By the direct calculations, we have $\pi \cdot \delta_1 = 0$ and $\delta_q \cdot \delta_{q+1} = 0$ $(q \geq 1)$.

To see that the complex (Y, π, δ) is acyclic, we state a contracting homotopy. In general, it suffices to define the homotopy as an abelian group homomorphism. However, we can see that there exists a homotopy as a right Γ-module, which permits us to cut down the number of cases. We define right Γ-homomorphisms $T_{-1} : \Gamma \to Y_0$ and $T_q : Y_q \to Y_{q+1}$ $(q \geq 0)$ as follows:

$$T_{-1}(\gamma) = c_0^1 \gamma \qquad (\text{for } \gamma \in \Gamma).$$

If $q (\geq 0)$ is even, then

$$T_q(\zeta^k c_q^s) = \begin{cases} 0 & (k = 0,\ s = 1), \\ \displaystyle\sum_{l=0}^{k-1} \zeta^{k-1-l} c_{q+1}^1 \zeta^l & (1 \leq k < t,\ s = 1), \\ 0 & (s (\geq 2) \text{ even}), \\ -\zeta^k c_{q+1}^{s+1} & (s (\geq 3) \text{ odd}), \end{cases}$$

$$T_q(\zeta^k j c_q^s) = \begin{cases} (-1)^{q/2} c_{q+1}^2 j & (k = 0,\ s = 1), \\ (-1)^{q/2} \left(\displaystyle\sum_{l=0}^{k-1} \zeta^{k-1-l} c_{q+1}^1 \zeta^l j + \zeta^k c_{q+1}^2 j \right) & (1 \leq k < t,\ s = 1), \\ \zeta^k j c_{q+1}^{s+1} & (s (\geq 2) \text{ even}), \\ 0 & (s (\geq 3) \text{ odd}). \end{cases}$$

If $q(\geq 1)$ is odd, then

$$T_q(\zeta^k c_q^s) = \begin{cases} 0 & (0 \leq k \leq t-2,\ s=1), \\ c_{q+1}^1 & (k=t-1,\ s=1), \\ 0 & (s(\geq 2) \text{ even}), \\ -\zeta^k c_{q+1}^{s+1} & (s(\geq 3) \text{ odd}), \end{cases}$$

$$T_q(\zeta^k j c_q^s) = \begin{cases} (-1)^{(q-1)/2}\left(c_{q+1}^1 j\zeta + \zeta^{t-1} c_{q+1}^2 j\zeta\right) & (k=0,\ s=1), \\ (-1)^{(q+1)/2}\zeta^{k-1} c_{q+1}^2 j\zeta & (1 \leq k < t,\ s=1), \\ \zeta^k j c_{q+1}^{s+1} & (s(\geq 2) \text{ even}), \\ 0 & (s(\geq 3) \text{ odd}). \end{cases}$$

Then by the direct calculations, we have

$$\delta_{q+1} T_q + T_{q-1}\delta_q = \mathrm{id}_{Y_q}$$

for $q \geq 0$. Hence (Y, π, δ) is a Γ^e-projective resolution of Γ. □

2 Hochschild cohomology $HH^*(\Gamma)$

2.1 Module structure

In this section, we give the module structure of $HH^*(\Gamma)$. This is obtained by using the Γ^e-projective resolution (Y, π, δ) of Γ stated in Theorem 1. In the following we denote a direct sum of q copies of a module M by M^q.

First, we state the following lemma:

Lemma 1. *Let ζ be a primitive 2^{r+1}-th root of 1 for any positive integer $r \geq 2$ and K the maximal real subfield $\mathbb{Q}(\zeta + \zeta^{-1})$ of $\mathbb{Q}(\zeta)$. Then $(\zeta + \zeta^{-1})^2$ divides 2 in R, where R denotes $\mathbb{Z}[\zeta + \zeta^{-1}]$, the ring of integers of K.*

Proof. See [4, Lemma 1]. Note that $\zeta^{2^k} + \zeta^{-2^k}$ divides 2 in R for $0 \leq k \leq r-2$. □

If $r \geq 2$, we set $\eta_k = 2e/(\zeta^{2^k} + \zeta^{-2^k})$ for $0 \leq k \leq r-2$ in the following. Let $\eta = \eta_0$.

In the following, we show that $e - \eta^2$ is an unit in R. If $r = 2$, then we have $e - \eta^2 = -e$. If $r \geq 3$, then we have

$$-(e-\eta^2)\prod_{k=1}^{r-2}(e+\eta_k)^2 = -(e-\eta_{r-2}^2) = e,$$

because the equation $(e-\eta_{k-1}^2)(e+\eta_k)^2 = e-\eta_k^2$ holds for $1 \leq k \leq r-2$. Therefore $e-\eta^2$ is an unit in R.

As elements of Γ^{q+1}, we set

$$\iota_q^s = \begin{cases} (0, \ldots, 0, \overset{s}{\check{e}}, 0, \ldots, 0) & (\text{if } 1 \leq s \leq q+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have $\Gamma^{q+1} = \bigoplus_{k=1}^{q+1} \Gamma \iota_q^k$.

Applying the functor $\mathrm{Hom}_{\Gamma^e}(-,\Gamma)$ to the resolution (Y,π,δ), we have the following complex, where we identify $\mathrm{Hom}_{\Gamma^e}(Y_q,\Gamma)$ with Γ^{q+1} using an isomorphism $\mathrm{Hom}_{\Gamma^e}(Y_q,\Gamma) \to \Gamma^{q+1}; f \mapsto \sum_{k=1}^{q+1} f(c_q^k)\iota_q^k$:

$$\left(\mathrm{Hom}_{\Gamma^e}(Y,\Gamma),\delta^{\#}\right): \quad 0 \to \Gamma \xrightarrow{\delta_1^{\#}} \Gamma^2 \xrightarrow{\delta_2^{\#}} \Gamma^3 \xrightarrow{\delta_3^{\#}} \Gamma^4 \xrightarrow{\delta_4^{\#}} \Gamma^5 \to \cdots,$$

$$\delta_{q+1}^{\#}(\gamma\iota_q^s) = \begin{cases} -\sum_{l=0}^{t-1} \zeta^{t-1-l}\gamma\zeta^l\iota_{q+1}^s + ((-1)^{(q-s)/2}\zeta j\gamma j\zeta + \gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ even, } s \text{ even,} \\ (\zeta\gamma - \gamma\zeta)\iota_{q+1}^s + ((-1)^{(q-s-1)/2} j\gamma j - \gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ even, } s \text{ odd,} \\ -(\zeta\gamma - \gamma\zeta)\iota_{q+1}^s + ((-1)^{(q-s-1)/2} j\gamma j + \gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ odd, } s \text{ even,} \\ \sum_{l=0}^{t-1} \zeta^{t-1-l}\gamma\zeta^l\iota_{q+1}^s + ((-1)^{(q-s)/2}\zeta j\gamma j\zeta - \gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ odd, } s \text{ odd.} \end{cases}$$

In the above, note that

$$\gamma\iota_q^s = \begin{cases} (0,\dots,0,\overset{s}{\check{\gamma}},0,\dots,0) & (\text{if } 1 \le s \le q+1), \\ 0 & (\text{otherwise}), \end{cases}$$

for $\gamma \in \Gamma$, and so on.

Theorem 2. (1) *If $r = 1$, the $\mathbb{Z}$-module structure of $HH^n(\Gamma)$ is given as follows:*

(i) *If $n = 0$, then $HH^0(\Gamma) = \mathbb{Z}$.*

(ii) *If $n = 1$, then $HH^1(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^3$ with generators $\zeta j\iota_1^1,\ j\iota_1^1 + \zeta j\iota_1^2,\ \zeta\iota_1^2$.*

(iii) *If $n = 2$, then $HH^2(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^5$ with generators $\zeta\iota_2^1,\ \iota_2^1 + \zeta\iota_2^2,\ j\iota_2^2,\ \zeta j\iota_2^2 - j\iota_2^3,\ \iota_2^3$.*

(iv) *If $n = 3$, then $HH^3(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^7$ with generators $j\iota_3^1,\ \zeta j\iota_3^1 - j\iota_3^2,\ \iota_3^2,\ \zeta\iota_3^2 - \iota_3^3,\ \zeta j\iota_3^3,\ j\iota_3^3 + \zeta j\iota_3^4,\ \zeta\iota_3^4$.*

(v) *If $n = 4k$ $(k \neq 0)$, then $HH^n(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1}$ with generators*

$$\iota_n^{4l+1},\ \zeta\iota_n^{4l+1} - \iota_n^{4l+2},\ \zeta j\iota_n^{4l+2},\ j\iota_n^{4l+2} + \zeta j\iota_n^{4l+3},\ \zeta\iota_n^{4l+3},\ \iota_n^{4l+3} + \zeta\iota_n^{4l+4},$$
$$j\iota_n^{4l+4},\ \zeta j\iota_n^{4l+4} - j\iota_n^{4l+5},\ \iota_n^{4k+1},$$

where $l = 0,1,2,\dots,k-1$.

(vi) *If $n = 4k+1$ $(k \neq 0)$, then $HH^n(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1}$ with generators*

$$\zeta j\iota_n^{4l+1},\ j\iota_n^{4l+1} + \zeta j\iota_n^{4l+2},\ \zeta\iota_n^{4l+2},\ \iota_n^{4m+2} + \zeta\iota_n^{4m+3},\ j\iota_n^{4m+3},$$
$$\zeta j\iota_n^{4m+3} - j\iota_n^{4m+4},\ \iota_n^{4m+4},\ \zeta\iota_n^{4m+4} - \iota_n^{4m+5},$$

where $l = 0,1,2,\dots,k$ and $m = 0,1,2,\dots,k-1$.

(vii) *If* $n = 4k+2$ $(k \neq 0)$, *then* $HH^n(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1}$ *with generators*

$$\zeta\iota_n^{4l+1},\ \iota_n^{4l+1} + \zeta\iota_n^{4l+2},\ j\iota_n^{4l+2},\ \zeta j\iota_n^{4l+2} - j\iota_n^{4l+3},\ \iota_n^{4l+3},$$
$$\zeta\iota_n^{4m+3} - \iota_n^{4m+4},\ \zeta j\iota_n^{4m+4},\ j\iota_n^{4m+4} + \zeta j\iota_n^{4m+5},$$

where $l = 0, 1, 2, \dots, k$ *and* $m = 0, 1, 2, \dots, k-1$.

(viii) *If* $n = 4k+3$ $(k \neq 0)$, *then* $HH^n(\Gamma) = (\mathbb{Z}/2\mathbb{Z})^{2n+1}$ *with generators*

$$j\iota_n^{4l+1},\ \zeta j\iota_n^{4l+1} - j\iota_n^{4l+2},\ \iota_n^{4l+2},\ \zeta\iota_n^{4l+2} - \iota_n^{4l+3},\ \zeta j\iota_n^{4l+3},$$
$$j\iota_n^{4l+3} + \zeta j\iota_n^{4l+4},\ \zeta\iota_n^{4l+4},\ \iota_n^{4m+4} + \zeta\iota_n^{4m+5},$$

where $l = 0, 1, 2, \dots, k$ *and* $m = 0, 1, 2, \dots, k-1$.

(2) *If* $r \geq 2$, *the* R*-module structure of* $HH^n(\Gamma)$ *is as follows:*

(i) *If* $n = 0$, *then* $HH^0(\Gamma) = R$.

(ii) *If* $n = 1$, *then* $HH^1(\Gamma) = (R/(\zeta + \zeta^{-1})R)^3$ *with generators* $(j - \eta\zeta j)\iota_1^1$, $(\zeta j - \eta j)\iota_1^1 + (j - \eta\zeta j)\iota_1^2$, $(e - \eta\zeta)\iota_1^2$.

(iii) *If* $n = 2$, *then* $HH^2(\Gamma) = R/2^rR \oplus (R/(\zeta + \zeta^{-1})R)^4$, *where the* $R/2^rR$ *summand is generated by* $(e - \eta\zeta)\iota_2^1$ *and the* $(R/(\zeta + \zeta^{-1})R)^4$ *summands are generated by* $2^{r-1}\eta\zeta\iota_2^1 + \zeta\iota_2^2$, $j\iota_2^2$, $\zeta j\iota_2^2 - j\iota_2^3$, ι_2^3.

(iv) *If* $n = 3$, *then* $HH^3(\Gamma) = (R/(\zeta + \zeta^{-1})R)^7$ *with generators* $j\iota_3^1$, $\zeta j\iota_3^1 - j\iota_3^2$, ι_3^2, $2^{r-1}\eta\zeta\iota_3^2 + (\zeta - \eta)\iota_3^3$, $(j - \eta\zeta j)\iota_3^3$, $(\zeta j - \eta j)\iota_3^3 + (j - \eta\zeta j)\iota_3^4$, $(e - \eta\zeta)\iota_3^4$.

(v) *If* $n = 4k$ $(k \neq 0)$, *then* $HH^n(\Gamma) = R/2^rR \oplus (R/(\zeta + \zeta^{-1})R)^{2n}$, *where the* $R/2^rR$ *summand is generated by* ι_n^1 *and the* $(R/(\zeta + \zeta^{-1})R)^{2n}$ *summands are generated by*

$$2^{r-1}\eta\zeta\iota_n^{4l+1} + (\zeta - \eta)\iota_n^{4l+2},\ (j - \eta\zeta j)\iota_n^{4l+2},\ (\zeta j - \eta j)\iota_n^{4l+2} + (j - \eta\zeta j)\iota_n^{4l+3},$$
$$(e - \eta\zeta)\iota_n^{4l+3},\ 2^{r-1}\eta\zeta\iota_n^{4l+3} + \zeta\iota_n^{4l+4},\ j\iota_n^{4l+4},\ \zeta j\iota_n^{4l+4} - j\iota_n^{4l+5},\ \iota_n^{4l+5},$$

where $l = 0, 1, 2, \dots, k-1$.

(vi) *If* $n = 4k+1$ $(k \neq 0)$, *then* $HH^n(\Gamma) = (R/(\zeta + \zeta^{-1})R)^{2n+1}$ *with generators*

$$(j - \eta\zeta j)\iota_n^{4l+1},\ (\zeta j - \eta j)\iota_n^{4l+1} + (j - \eta\zeta j)\iota_n^{4l+2},\ (e - \eta\zeta)\iota_n^{4l+2},$$
$$2^{r-1}\eta\zeta\iota_n^{4m+2} + \zeta\iota_n^{4m+3},\ j\iota_n^{4m+3},\ \zeta j\iota_n^{4m+3} - j\iota_n^{4m+4},\ \iota_n^{4m+4},$$
$$2^{r-1}\eta\zeta\iota_n^{4m+4} + (\zeta - \eta)\iota_n^{4m+5},$$

where $l = 0, 1, 2, \dots, k$ *and* $m = 0, 1, 2, \dots, k-1$.

(vii) *If* $n = 4k+2$ $(k \neq 0)$, *then* $HH^n(\Gamma) = R/2^rR \oplus (R/(\zeta + \zeta^{-1})R)^{2n}$, *where the* $R/2^rR$ *summand is generated by* $(e - \eta\zeta)\iota_n^1$ *and the* $(R/(\zeta + \zeta^{-1})R)^{2n}$ *summands are generated by*

$$2^{r-1}\eta\zeta\iota_n^{4l+1} + \zeta\iota_n^{4l+2},\ j\iota_n^{4l+2},\ \zeta j\iota_n^{4l+2} - j\iota_n^{4l+3},\ \iota_n^{4l+3},\ 2^{r-1}\eta\zeta\iota_n^{4m+3} + (\zeta - \eta)\iota_n^{4m+4},$$
$$(j - \eta\zeta j)\iota_n^{4m+4},\ (\zeta j - \eta j)\iota_n^{4m+4} + (j - \eta\zeta j)\iota_n^{4m+5},\ (e - \eta\zeta)\iota_n^{4m+5},$$

where $l = 0, 1, 2, \dots, k$ *and* $m = 0, 1, 2, \dots, k-1$.

(viii) *If* $n = 4k+3$ $(k \neq 0)$, *then* $HH^n(\Gamma) = (R/(\zeta+\zeta^{-1})R)^{2n+1}$ *with generators*

$$j\iota_n^{4l+1},\ \zeta j\iota_n^{4l+1} - j\iota_n^{4l+2},\ \iota_n^{4l+2},\ 2^{r-1}\eta\zeta\iota_n^{4l+2} + (\zeta-\eta)\iota_n^{4l+3},\ (j-\eta\zeta j)\iota_n^{4l+3},$$
$$(\zeta j - \eta j)\iota_n^{4l+3} + (j-\eta\zeta j)\iota_n^{4l+4},\ (e-\eta\zeta)\iota_n^{4l+4},\ 2^{r-1}\eta\zeta\iota_n^{4m+4} + \zeta\iota_n^{4m+5},$$

where $l = 0,1,2,\dots,k$ *and* $m = 0,1,2,\dots,k-1$.

Proof. The proof is straightforward. However it is complicated. □

2.2 Ring structure

In this subsection, we will determine the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$.

Recall the Yoneda product in $HH^*(\Gamma)$. Let $\alpha \in HH^n(\Gamma)$ and $\beta \in HH^m(\Gamma)$, where α and β are represented by cocycles $f_\alpha : Y_n \to \Gamma$ and $f_\beta : Y_m \to \Gamma$, respectively. There exists the commutative diagram of Γ^e-modules:

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_{n+m+1}"] & Y_{n+m} \arrow[r, "\delta_{n+m}"] \arrow[d, "\mu_n"] & \cdots \arrow[r, "\delta_{m+2}"] & Y_{m+1} \arrow[r, "\delta_{m+1}"] \arrow[d, "\mu_1"] & Y_m \arrow[r, "f_\beta"] \arrow[d, "\mu_0"] & \Gamma \arrow[d, equal] & \\
\cdots \arrow[r, "\delta_{n+1}"'] & Y_n \arrow[r, "\delta_n"'] & \cdots \arrow[r, "\delta_2"'] & Y_1 \arrow[r, "\delta_1"'] & Y_0 \arrow[r, "\pi"'] & \Gamma \arrow[r] & 0,
\end{tikzcd}$$

where μ_l $(0 \le l \le n)$ are liftings of f_β. We define the product $\alpha\cdot\beta \in HH^{n+m}(\Gamma)$ by the cohomology class of $f_\alpha\mu_n$. This product is independent of the choice of representatives f_α and f_β, and liftings μ_l $(0 \le l \le n)$.

First, we consider the case $r = 1$. Note the Hochschild cohomology ring $HH^*(\Gamma)$ is graded-commutative. From Theorem 2 (1), $HH^*(\Gamma)$ is a commutative ring in this case. We take generators of $HH^1(\Gamma)$ as follows:

$$A = \zeta\iota_1^2,\ B = \zeta j\iota_1^1,\ C = j\iota_1^1 + \zeta j\iota_1^2.$$

Then we have $2A = 2B = 2C = 0$. We calculate the Yoneda products. Then $HH^n(\Gamma)$ $(n \ge 2)$ is multiplicatively generated by A, B and C, and the equation $A^2+B^2+C^2 = 0$ holds. Moreover the relations are enough. Thus we can determine the ring structure of $HH^*(\Gamma)$ in the case $r = 1$ (see [3, Section 3.1] for details).

Next, we consider the case $r \ge 2$. The computation is similar to the case where $r = 1$, however it is more complicated. By Theorem 2 (2), we take generators of $HH^1(\Gamma)$ as follows:

$$A = (e-\eta\zeta)\iota_1^2,\ B = (j-\eta\zeta j)\iota_1^1,\ C = (\zeta j - \eta j)\iota_1^1 + (j-\eta\zeta j)\iota_1^2.$$

Then we have $(\zeta+\zeta^{-1})A = (\zeta+\zeta^{-1})B = (\zeta+\zeta^{-1})C = 0$. Note that products of A, B, C and $X \in HH^n(\Gamma)$ $(n \ge 0)$ are commutative, because $HH^*(\Gamma)$ is graded-commutative and the equations $2A = 2B = 2C = 0$ hold. By calculating the Yoneda products we have the following proposition.

Proposition 2. *If $r \geq 2$, then the following equations hold in $HH^2(\Gamma)$:*

$$A^2 = \iota_2^3,\ AB = j\iota_2^2,\ AC = \zeta j\iota_2^2 - j\iota_2^3,\ B^2 = 2^{r-1}\eta\zeta\iota_2^1 + \zeta\iota_2^2,$$
$$BC = 2^{r-1}\eta(e - \eta\zeta)\iota_2^1,\ C^2 = 2^{r-1}\eta\zeta\iota_2^1 + \zeta\iota_2^2 + \iota_2^3.$$

In particular, generators of $HH^2(\Gamma)$ except $(e - \eta\zeta)\iota_2^1$ are generated by the products of A, B and C, and the equation $A^2 + B^2 + C^2 = 0$ holds.

In the following, we put $D = (e - \eta\zeta)\iota_2^1$ which is a generator of $HH^2(\Gamma)$, and then we have $2^r D = 0$ and $BC = 2^{r-1}\eta D$. Similarly, we calculate the Yoneda products. Then $HH^n(\Gamma)$ $(n \geq 3)$ is multiplicatively generated by A, B, C and D, and the relations are enough. Thus we can determine the ring structure of $HH^*(\Gamma)$ in the case $r \geq 2$ (see [3, Section 3.2] for details).

We state the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$ by summarizing these computations.

Theorem 3. (1) *If $r = 1$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is isomorphic to*

$$\mathbb{Z}[A, B, C]/(2A, 2B, 2C, A^2 + B^2 + C^2),$$

where $\deg A = \deg B = \deg C = 1$.

(2) *If $r \geq 2$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is isomorphic to*

$$R[A, B, C, D]/((\zeta + \zeta^{-1})A, (\zeta + \zeta^{-1})B, (\zeta + \zeta^{-1})C, 2^r D,$$
$$A^2 + B^2 + C^2, BC - 2^{r-1}\eta D),$$

where $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, $\deg A = \deg B = \deg C = 1$ *and* $\deg D = 2$.

Remark. In the case $r = 1$, this cohomology ring is already known by Sanada [8, Section 3.4]. In [8], he treats the Hochschild cohomology of crossed products over a commutative ring and its product structure using a spectral sequence of a double complex. As a special case, he determines the Hochschild cohomology ring of the quaternion algebra over $\mathbb{Z}$.

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